$(1+x^2)y'' + 5xy' + 3y = 0$ Diferencijalna jednadžba (DJ)

Rješenje tražimo u obliku

$$y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots = \sum_{n=0}^{\infty} a_n \cdot x^n \qquad (1)$$

gdje a_i su neodređeni $i \in \{0, 1, 2, \dots\}$

Zadatak će biti riješen kada:

1) Nađemo a_i, zamjenjujući (1) i njegove y' i y'' u DJ.

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots + n a_n x^{n-1} + \dots = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = 2a_2 + 3 \cdot 2 a_3 x + \dots + n(n-1) a_n x^{n-2} + \dots = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

2) Provjerimo dali dobiveni red je konvergentan u nekoj oblasti.

$$(1+x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 5x\sum_{n=1}^{\infty} n a_n x^{n-1} + 3\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} 5n a_n x^n + \sum_{n=0}^{\infty} 3a_n x^n = 0$$

$$\underline{2\cdot 1 a_2} + \underline{3\cdot 2 a_3 x} + 4\cdot 3 a_4 x^2 + 5\cdot 4 a_5 x^3 + 6\cdot 5\cdot a_6 x^4 + 7\cdot 6\cdot a_7 x^5 + \dots +$$
$$+ \quad 2\cdot 1 a_2 x^2 + 3\cdot 2 a_3 x^3 + 4\cdot 3 a_4 x^4 + 5\cdot 4 a_5 x^5 + \dots +$$
$$+ \quad \underline{5\cdot 1 a_1 x} + 5\cdot 2 a_2 x^2 + 5\cdot 3 a_3 x^3 + 5\cdot 4 a_4 x^4 + 5\cdot 5 a_5 x^5 + \dots +$$
$$+ \underline{3a_0} + \underline{3a_1 x} + 3a_2 x^2 + 3a_3 x^3 + 3a_4 x^4 + 3a_5 x^5 + \dots = 0$$

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ispred slobodnog ČLANA: $2a_2 + 3a_0 = 0 \Rightarrow a_2 = -\frac{3}{2}a_0$

ispred x: $3\cdot 2 a_3 + 5\cdot 1 a_1 + 3a_1 = 0 \Rightarrow a_3 = -\frac{4}{3}a_1$

x^2: $4\cdot 3 a_4 + 2\cdot 1 a_2 + 5\cdot 2 a_2 + 3a_2 = 0 \Rightarrow 12a_4 + 15a_2 = 0 \Rightarrow a_4 = -\frac{5}{4}a_2$

x^3: $5\cdot 4 a_5 + 3\cdot 2 a_3 + 5\cdot 3 a_3 + 3a_3 = 0$

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x^n: $(n+2)(n+1)a_{n+2} + \underbrace{5n a_n}_{5n a_n} + n(n-1)a_n + 3a_n = 0$

$$(n+2)(n+1)a_{n+2} + a_n(5n + n^2 - n + 3) = 0$$

$$(n+2)(n+1)a_{n+2} + a_n(n^2 + 4n + 3) = 0$$

$$(n+2)(n+1)a_{n+2} + a_n(n+3)(n+1) = 0 \;/:(n+1) \;;\; n \neq -1$$

$$\boxed{(n+2)a_{n+2} + a_n(n+3) = 0}$$

t.j. $\boxed{(n+2)a_{n+2} = -(n+3)a_n} \Rightarrow \underline{a_{n+2} = -\frac{n+3}{n+2}a_n}$

$a_0 = a_0$

$a_1 = a_1$

$a_2 = -\frac{3}{2}a_0$

$a_3 = -\frac{4}{3}a_1$

$a_4 = -\frac{5}{4}\cdot\left(-\frac{3}{2}\right)a_0 = \frac{5\cdot 3}{4\cdot 2}a_0$

$a_5 = -\frac{6}{5}a_3 = -\frac{6}{5}\cdot\left(-\frac{4}{3}\right)a_1 = \frac{6\cdot 4}{5\cdot 3}a_1$

$a_6 = -\frac{7}{6}a_4 = -\frac{7}{6}\cdot\frac{5\cdot 3}{4\cdot 2}a_0 = -\frac{7\cdot 5\cdot 3}{6\cdot 4\cdot 2}a_0$

$a_7 = -\frac{8}{7}a_5 = -\frac{8}{7}\cdot\frac{6\cdot 4}{5\cdot 3}a_1 = -\frac{8\cdot 6\cdot 4}{7\cdot 5\cdot 3}a_1$

$a_8 = -\frac{9}{8}a_6 = -\frac{9}{8}\cdot\left(-\frac{7\cdot 5\cdot 3}{6\cdot 4\cdot 2}a_0\right) = \frac{9\cdot 7\cdot 5\cdot 3}{8\cdot 6\cdot 4\cdot 2}a_0$

Vidimo da ćemo sve dobiti pomoću koeficijenata a_0, a_1. $a_0, a_2, a_4, a_6, a_8, \dots$ preko a_0
$a_1, a_3, a_5, a_7, a_9, \dots$ preko a_1.

Da vidimo šta imamo sada za rješenje:

$$y = a_0 + a_1 x - \frac{3}{2} a_0 x^2 - \frac{4}{3} a_1 x^3 + \frac{5\cdot 3}{4\cdot 2} a_0 x^4 + \frac{6\cdot 4}{5\cdot 3} a_1 x^5 - \frac{7\cdot 5\cdot 3}{6\cdot 4\cdot 2} a_0 x^6 - \frac{8\cdot 6\cdot 4}{7\cdot 5\cdot 3} a_1 x^7 +$$

$$+ \frac{9\cdot 7\cdot 5\cdot 3}{8\cdot 6\cdot 4\cdot 2} a_0 \cdot x^8 + \frac{10\cdot 8\cdot 6\cdot 4}{9\cdot 7\cdot 5\cdot 3} x^9 + \dots + \dots$$

$$y = a_0\left(1 - \frac{3}{2}x^2 + \frac{5\cdot 3}{4\cdot 2}x^4 - \frac{7\cdot 5\cdot 3}{6\cdot 4\cdot 2}x^6 + \frac{9\cdot 7\cdot 5\cdot 3}{8\cdot 6\cdot 4\cdot 2}x^8 - \dots\right) +$$

$$+ a_1\left(x - \frac{4}{3}x^3 + \frac{6\cdot 4}{5\cdot 3}x^5 - \frac{8\cdot 6\cdot 4}{7\cdot 5\cdot 3}x^7 + \frac{10\cdot 8\cdot 6\cdot 4}{9\cdot 7\cdot 5\cdot 3}x^9 - \dots \frac{\cdot 2}{\cdot 2}\right)$$

množimo i gore i dolje zbog faktorijela!

$$\boxed{y = a_0 \cdot \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)!!}{(2n)!!} x^{2n}}_{f(x)} + a_1 \frac{x}{2} \cdot \underbrace{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n)!!}{(2n-1)!!} x^{2n-2}}_{g(x)}}$$

$$\boxed{y = a_0 \cdot f(x) + a_1 \cdot \frac{x}{2} g(x)}$$