(2) $w=f(z)=u+iv$ ako $u(x,y)=\varphi\left(\frac{x^2+y^2}{y}\right)$ smjena $t=\frac{x^2+y^2}{y}$

Caushy-Riemann uvjeti

$$\begin{cases} \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \end{cases}$$

i LAPLASOVA PARCIJALNA DIFERENCIJALNA JEDNADŽBA

$$\boxed{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0} \quad i \quad \boxed{\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0}$$

OVO ĆE NAM KORISTITI

VAŽE ZA ANALITIČKU FUNKCIJU $f(z)=u+iv$

$$\frac{1}{t}=\frac{y}{x^2+y^2}$$

$$t'_x=\frac{2x}{y}$$

$$t'_y=-\frac{x^2}{y^2}+1=\frac{y^2-x^2}{y^2}$$

DERIVACIJA SLOŽENE FUNKCIJE

$$[\varphi(t)]'_x=\varphi'(t)\cdot t'_x$$

Počnimo,

$$\frac{\partial u}{\partial x}=\underbrace{\varphi'}_{u}\cdot\underbrace{\frac{2x}{y}}_{v} \Rightarrow \frac{\partial^2 u}{\partial x^2}=\underbrace{\left(\varphi''\cdot\frac{2x}{y}\right)}_{u'}\cdot\underbrace{\frac{2x}{y}}_{v}+\underbrace{\varphi'}_{u}\cdot\underbrace{\frac{2}{y}}_{v'}=\frac{4x^2}{y^2}\cdot\varphi''+\frac{2}{y}\cdot\varphi'$$

$$\frac{\partial u}{\partial y}=\varphi'\cdot\frac{y^2-x^2}{y^2}=\underbrace{\varphi'}_{u}\cdot\underbrace{\left(1-\frac{x^2}{y^2}\right)}_{v} \Rightarrow \frac{\partial^2 u}{\partial y^2}=\underbrace{\left[\varphi''\cdot\left(1-\frac{x^2}{y^2}\right)\right]}_{u'}\underbrace{\left(1-\frac{x^2}{y^2}\right)}_{v}+\underbrace{\varphi'}_{u}\cdot\underbrace{\frac{2x^2}{y^3}}_{v'}=$$

$$\boxed{\frac{\partial^2 u}{\partial y^2}=\left(\frac{y^2-x^2}{y^2}\right)^2\cdot\varphi''+\frac{2x^2}{y^3}\cdot\varphi'} \quad i \quad \boxed{\frac{\partial^2 u}{\partial x^2}=\frac{4x^2}{y^2}\cdot\varphi''+\frac{2}{y}\cdot\varphi'}$$

$+$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\left[\frac{4x^2}{y^2}+\frac{(y^2-x^2)^2}{y^4}\right]\cdot\varphi''+\left(\frac{2}{y}+\frac{2x^2}{y^3}\right)\cdot\varphi'\overset{\text{II}}{=}0$$

$$\frac{4x^2y^2+y^4-2x^2y^2+x^4}{y^4}\cdot\varphi''+\frac{2y^2+2x^2}{y^3}\cdot\varphi'=0$$

$$\frac{(x^2+y^2)^2}{y^4}\cdot\varphi''+\frac{2(x^2+y^2)}{y^3}\cdot\varphi'=0 \overset{:\frac{x^2+y^2}{y^3}}{\Longrightarrow} \frac{(x^2+y^2)}{y}\cdot\varphi''+2\varphi'=0$$

$$\varphi''+\frac{2y}{(x^2+y^2)}\varphi'=0 \Rightarrow \boxed{\varphi''+\frac{2}{t}\varphi'=0}$$

DIFERENCIJALNA JEDNADŽ.

neka $\varphi'=f$, $\varphi''=f'$ $\Rightarrow f'=-\frac{2}{t}f$

$$\frac{df}{dt}=-\frac{2}{t}f \Rightarrow \frac{df}{f}=-\frac{2dt}{t} \Big/\int$$

$$\int\frac{df}{f}=-2\int\frac{dt}{t}$$

$$\ln f=-2\ln t+\ln c_1$$

$$\ln f=\ln\frac{c_1}{t^2}$$

$$f=\frac{c_1}{t^2}=\varphi'$$

$$\frac{d\varphi}{dt}=\frac{c_1}{t^2}\Rightarrow d\varphi=c_1\frac{dt}{t^2}\Big/\int$$

$$\boxed{\varphi=-\frac{c_1}{t}+c_2}$$

Imamo dakle: $\varphi(t)=-\frac{c_1}{t}+c_2$

$$\boxed{u(x,y)=\varphi\left(\frac{x^2+y^2}{y}\right)=-\frac{c_1 y}{x^2+y^2}+c_2}$$

Ovim smo našli REALNI DIO KOMPLEKSNE FUNKCIJE $w=f(z)$

Sada idemo standardnim rješavanjem zadatka koristeći Caushy-Riemann uvjete :)

$$u(x,y) = -\frac{C_1 \cdot y}{x^2+y^2} + C_2 \Rightarrow \frac{\partial u}{\partial y} = -\frac{C_1(x^2+y^2) - C_1 y \cdot 2y}{(x^2+y^2)^2} = \frac{C_1(y^2-x^2)}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial x} = -\frac{0\cdot(x^2+y^2) - C_1 y \cdot 2x}{(x^2+y^2)^2} = -\frac{2C_1 xy}{(x^2+y^2)^2} = \frac{\partial v}{\partial y} \Rightarrow V = -2C_1 \int \frac{xy}{(x^2+y^2)^2}dy + \psi(x)$$

$$x^2+y^2 = t \;/'_y$$
$$2y\,dy = dt$$

$$V = -\cancel{2}C_1 \cdot \int \frac{x \cdot \frac{dt}{\cancel{2}}}{t^2} + \psi(x) = -C_1 x \int \frac{dt}{t^2} + \psi(x)$$

$$V = \frac{C_1 x}{t} + \psi(x) = \frac{C_1 x}{x^2+y^2} + \psi(x) = V$$

odovoga nađimo $\frac{\partial v}{\partial x}$

$$\frac{\partial v}{\partial x} = \frac{C_1\cdot(x^2+y^2) - C_1 x \cdot 2x}{(x^2+y^2)^2} + \psi'(x) = -\frac{\partial u}{\partial y}$$

$$\frac{C_1(y^2-x^2)}{(x^2+y^2)^2} + \psi'(x) = -\frac{C_1(y^2-x^2)}{(x^2+y^2)^2}$$

$$(+2x^2-2x^2) \quad \psi'(x) = -\frac{2C_1(y^2-x^2)}{(x^2+y^2)^2} \Big/ \int$$

$$\psi(x) = -2C_1 \cdot \int \frac{y^2-x^2}{(x^2+y^2)^2}dx + C = -2C_1 \cdot \int \left[\frac{y^2-x^2+2x^2}{(x^2+y^2)^2} - \frac{2x^2}{(x^2+y^2)^2}\right]dx =$$

$$= -2C_1 \cdot \int\left[\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}\right]dx = -2C_1 \cdot \frac{1}{y}\operatorname{arctg}\frac{x}{y} + 4C_1 \cdot \underbrace{\int \frac{x^2\,dx}{(x^2+y^2)^2}}_{I_1} = (*)$$

$$\boxed{\int \frac{dx}{x^2+y^2} = \frac{1}{y}\cdot \operatorname{arctg}\frac{x}{y} + C}$$

TABLIČNI

$$I_1 = \int \frac{x^2\,dx}{(x^2+y^2)^2} = \int x \cdot \frac{x\,dx}{(x^2+y^2)^2} = -\frac{x}{2(x^2+y^2)} + \frac{1}{2}\int\frac{dx}{x^2+y^2} = -\frac{x}{2(x^2+y^2)} + \frac{1}{2y}\operatorname{arctg}\frac{x}{y} + C = I_1$$

$$u = x \Rightarrow du = dx$$

$$dv = \frac{x\,dx}{(x^2+y^2)^2} \Rightarrow v = \int\frac{x\,dx}{(x^2+y^2)^2} = \frac{1}{2}\int\frac{dt}{t^2} = -\frac{1}{2t} = -\frac{1}{2(x^2+y^2)}$$

$$x^2+y^2 = t \;/'_x$$
$$2x\,dx = dt$$
$$x\,dx = \frac{1}{2}dt$$

$$\boxed{\int u\cdot dv = u\cdot v - \int v\cdot du}$$

PARCIJALNA INTEGRACIJA

Konačno imamo:

$$\psi(x) = \cancel{-\frac{2C_1}{y}\operatorname{arctg}\frac{x}{y}} - \frac{2C_1 x}{x^2+y^2} + \cancel{\frac{2C_1}{y}\operatorname{arctg}\frac{x}{y}} + C$$

$$\boxed{\psi(x) = -\frac{2C_1 x}{x^2+y^2} + C}$$

Definitivno: $V(x,y) = \frac{C_1 x}{x^2+y^2} - \frac{2C_1 x}{x^2+y^2} + C$

$$\boxed{V(x,y) = -\frac{C_1 x}{x^2+y^2} + C}$$

$$W = f(z) = \overbrace{\left(-\frac{C_1 y}{x^2+y^2} + C_2\right)}^{u(x,y)} + i\cdot\overbrace{\left(-\frac{C_1 x}{x^2+y^2} + C\right)}^{v(x,y)}$$

Rješenje :)