Pođimo od jednačine $z^{2n}-1=0$ (1) $n\in\mathbb{N}$ koja ima $2n$-rješenja

Kako naći ta rješenja? $z^{2n}=1=\cos 2k\pi+i\sin 2k\pi$; $k=0,1,\dots 2n-1$

$$z_k=\sqrt[2n]{1}=\cos\frac{2k\pi}{2n}+i\sin\frac{2k\pi}{2n};\ k=0,1,\dots 2n-1$$

Primjetimo da za $k=0\Rightarrow z_0=1$ i to su jedina 2 realna korijena
za $k=n\Rightarrow z_n=-1$

Ostale korijene dobijamo za $k=1,2,3,\dots\dots,n-1$
i $k=n+1,n+2,n+3,\dots\dots,\underbrace{2n-1}_{n+(n-1)}$

Pošto smo našli $z_0=1$, $z_n=-1$ onda jednačinu možemo upisati.

$$z^{2n}-1=\underset{z_0=1}{(z-1)}\underset{z_n=-1}{(z+1)}\underbrace{\left(z^{2n-2}+z^{2n-4}+\dots+z^6+z^4+z^2+1\right)}_{\text{ovdje su } z_1,z_2,\dots z_{n-1} \text{ i } z_{n+1},z_{n+2},\dots z_{2n-1}}=0$$

Sada nas zanima ovaj „veliki polinom".

Neka $f(z)=1+z^2+z^4+\dots z^{2n-2}=\sum_{k=0}^{n-1}z^{2k}$

Najvažnije nam je da uspijemo dokazati da $\sum_{k=0}^{n-1}z^{2k}=\prod_{k=1}^{n-1}\left(z^2-2z\cdot\cos\frac{2k\pi}{2n}+1\right)$ ŠTO BAŠ i NIJE LAKO!

t.j. $f(z)=\sum_{k=0}^{n-1}z^{2k}=\prod_{k=1}^{n-1}\left(z^2-2z\cdot\cos\frac{k\pi}{2n}+1\right)$

Pođimo s dokazom. Jednačina $f(z)=0$ ima $2\cdot(n-1)=2n-2$ rješenja koja su konjugirano kompleksna. STVARNO:

$$z_k=\cos\frac{2k\pi}{2n}+i\sin\frac{2k\pi}{2n}\quad \text{za } k=1,2,3,4,\dots n-1,\ n+1,\ n+2,\dots,2n-1$$

Neka $k=j\Rightarrow z_j\cdot z_{2n-j}=\left(\cos\frac{2j\pi}{2n}+i\sin\frac{2j\pi}{2n}\right)\left(\cos\frac{2(2n-j)\pi}{2n}+i\sin\frac{2(2n-j)\pi}{2n}\right)=$
$j=1,2,3,\dots n-1$

$$=\left(\cos\frac{2j\pi}{2n}+i\sin\frac{2j\pi}{2n}\right)\left(\cos\left(2\pi-\frac{2j\pi}{2n}\right)+i\sin\left(2\pi-\frac{2j\pi}{2n}\right)\right)=$$

$$=\left(\cos\frac{2j\pi}{2n}+i\sin\frac{2j\pi}{2n}\right)\left(\cos\frac{2j\pi}{2n}+i\sin\left(-\frac{2j\pi}{2n}\right)\right)=$$

$$=\underbrace{\left(\cos\frac{2j\pi}{2n}+i\sin\frac{2j\pi}{2n}\right)}_{z_j}\underbrace{\left(\cos\frac{2j\pi}{2n}-i\sin\frac{2j\pi}{2n}\right)}_{\bar{z}_j}$$

$$\boxed{\begin{aligned}\cos(360^\circ-\alpha)&=\cos\alpha\\ \sin(360^\circ-\alpha)&=-\sin\alpha\end{aligned}}$$

$\bar{z}_j$ konjugirano $=z_{2n-j}$

Slično $z_j+z_{2n-j}=$
$=\cos\frac{2j\pi}{2n}+\cancel{i\sin\frac{2j\pi}{2n}}+\cos\frac{2j\pi}{2n}-\cancel{i\sin\frac{2j\pi}{2n}}$
$=\boxed{2\cos\frac{2j\pi}{2n}=z_j+z_{2n-j}}$

$$=\cos^2\frac{2j\pi}{2n}+\sin^2\frac{2j\pi}{2n}=\boxed{1}$$

Odavde možemo zaključiti da

$$f(z)=\prod_{j=1}^{n-1}(z-z_j)(z-z_{2n-j})=\prod_{j=1}^{n-1}\left(z^2-z(z_j+z_{2n-j})+z_j\cdot z_{2n-j}\right)=$$

umjesto $j\to k$

$$f(z)=\prod_{k=1}^{n-1}\left(z^2-2z\cdot\cos\frac{2k\pi}{2n}+1\right)$$ ŠTA SMO HTELI DOKAZATI

Sad je više lako. Uzmimo $z=1$

$$(2) \qquad f(1)=\sum_{k=0}^{n-1} 1^{2k} = \prod_{k=1}^{n-1}\left(\underset{1}{\cancel{z}}^2 - 2\underset{1}{\cancel{z}}\cdot\cos\frac{2k\pi}{2n}+1\right)$$

$$\underbrace{(1+1+1+\dots+1)}_{n\text{-puta}} = \prod_{k=1}^{n-1}\left(2-2\cos\frac{2k\pi}{2n}\right)$$

$$n = \prod_{k=1}^{n-1} 2\left(1-\cos\frac{2k\pi}{2n}\right)$$

koristimo: $\boxed{1-\cos\alpha = 2\sin^2\frac{\alpha}{2}}$

$$n = 2^{n-1}\cdot\prod_{k=1}^{n-1} 2\sin^2\frac{k\pi}{2n}$$

$$n = 2^{n-1}\cdot 2^{n-1}\cdot\left(\prod_{k=1}^{n-1}\sin\frac{k\pi}{2n}\right)^2$$

$$\frac{n}{2^{(n-1)\cdot 2}} = \left(\prod_{k=1}^{n-1}\sin\frac{k\pi}{2n}\right)^2 \quad / \sqrt{}$$

$$\boxed{\prod_{k=1}^{n-1}\sin\frac{k\pi}{2n} = \frac{\sqrt{n}}{2^{n-1}}}$$

ŠTA SMO TREBALI DOKAZATI